The distortion of knots

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Two views of the figure eight knot.

Origins of knot theory can be traced back to Lord Kelvin who thought different elements corresponded to different knots (1860s). First knot tables (1885) compiled by physicist Peter Guthrie Tait.
Knots

Five views of the unknot.

Definition

A knot is a map $\gamma : S^1 \to \mathbb{R}^3$ up to homotopy through smooth embeddings.
Knot table

Unknot

3₁

4₁

5₁

5₂

6₁

6₂

6₃

7₁

7₂

7₃

7₄

7₅

7₆

7₇
Distortion

Definition (Gromov)

The distortion of a (unit speed) curve \( \gamma : S^1 \to \mathbb{R}^3 \) is:

\[
\delta(\gamma) := \sup_{s,t \in S^1} \frac{|\gamma(s) - \gamma(t)|}{|s - t|} \geq 1
\]  

(1)

The distortion of a knot \( K \) is the minimal distortion among all curves \( \gamma \) in the knot class \( K \).

Numerical simulations indicate that \( \delta(\text{trefoil}) < 7.16 \).
Knots with small distortion

There are rather wild knots with finite distortion.

Curves with finite distortion can have infinite total curvature.

Thus distortion is a very weak measure of complexity.
Distortion of knots

Theorem (Gromov)
For any closed curve $\gamma$, we have $\delta(\gamma) \geq \frac{1}{2}\pi = 1.57 \ldots$. Equality holds if and only if $\gamma$ is a round circle.

Corollary (Gromov)
$\delta(\text{unknot}) = \frac{1}{2}\pi = 1.57 \ldots$.

Theorem (Denne–Sullivan)
For any knotted closed curve $\gamma$, we have $\delta(\gamma) \geq \frac{5}{3}\pi = 5.23 \ldots$.

Corollary (Denne–Sullivan)
$\delta(K) \geq \frac{5}{3}\pi = 5.23 \ldots$ for $K \neq \text{unknot}$. 
Torus knots

A torus and the torus knot $T_{3,7}$

**Lemma**

The standard embedding of the torus knot $T_{p,q}$ in $\mathbb{R}^3$ has distortion $\gg \max(p, q)$.

**Question (Gromov)**

Are there knots with arbitrarily large distortion? Specifically, is it true that $\delta(T_{p,q}) \to \infty$?
Distortion of torus knots

Theorem (P)

\[ \delta(T_{p,q}) \geq \frac{1}{160} \min(p, q) \text{ for torus knots } T_{p,q}. \]
Proof that torus knots have large distortion

Ingredient 1 (integral geometry):

$$
\int_{-\infty}^{\infty} \#(\gamma \cap H_t) \, dt \leq \text{length}(\gamma)
$$

(2)

where $H_t$ is the hyperplane $\{(x, y, z) \in \mathbb{R}^3 : z = t\}$.
Proof that torus knots have large distortion

Ingredient 2:
Let $\gamma \subseteq T \subseteq \mathbb{R}^3$ be the $(p, q)$-torus knot.
Given a family of balls $\{B_t\}_{t \in [0,1]}$ with $\#(\gamma \cap \partial B_t) < \min(p, q)$ for all $t \in [0,1]$, we have:

$$g(B_0 \cap T) = g(B_1 \cap T) \quad (3)$$

Key point: $\#(\gamma \cap \partial B_t) < \min(p, q)$ implies that $\partial B_t \cap T$ is inessential in $T$. 
Proof that torus knots have large distortion

Suppose $\gamma \subseteq T \subseteq \mathbb{R}^3$ with $\delta(\gamma) \ll \min(p, q)$.

Take any ball $B(r)$ of radius $r$ such that $g(B \cap T) = 1$.

**Ingredient 1** $\implies$ there exists $r \leq r' \leq \frac{11}{10} r$ such that 
$\#(\gamma \cap \partial B(r')) \ll \delta(\gamma) \ll \min(p, q)$. Similarly, find a disk cutting $B$ in half (approximately), and intersecting $\gamma$ in $\ll \min(p, q)$ places.

![Diagram of torus knots and ball intersections]

**Ingredient 2** $\implies$ $T$ intersected with upper or lower half-ball has genus 1.

We have thus produced a smaller ball with the same property $g(B' \cap T) = 1!$ (contradiction)
Questions

Question

Is it true that $\delta(T_2,p) \to \infty$ as $p \to \infty$?

L. Studer has shown that $\delta(T_2,p) \ll p/\log p$ (the standard embedding of $T_2,p$ has distortion $\asymp p$).

Question

Is it true that $\delta(T_p,q\#K) \to \infty$ as $p,q \to \infty$ (uniformly in $K$)?
Outlook

Topology $\leftrightarrow$ Geometry
manifolds, knots, etc. $\leftrightarrow$ curvature, distortion, etc.
Credits

Pictures taken from:

- Wikipedia
Thank you